Probability Proofs

Alex Houtz*

June 27, 2022

Properties of the Probability Measure

For each property that follows, I assume that $P(\cdot)$ is a probability measure defined on $\mathscr{F}$ and that events A and B are elements of $\mathscr{F}$.

Property 1. $P(A^c) = 1 - P(A)$

Proof. By definition, A and A^c are mutually disjoint and $A \cup A^c = \Omega$. By the second axiom of probability, we know that $P(\Omega) = 1$. Then:

$$\begin{aligned} P(\Omega) &= 1 \\ P(A \cup A^c) &= 1 && \text{(Now apply axiom 3:)} \\ P(A) + P(A^c) &= 1 \\ P(A^c) = 1 - P(A) \end{aligned}$$

We have now proven property 1. ■

Property 2. $P(\emptyset) = 0$

Proof. By the second axiom of probability, $P(\Omega) = 1$. Note that Ω and $\emptyset$ are disjoint.

*Math Camp 2022 Instructor | University of Notre Dame

In addition, recall that the probability measure has an upper bound of 1. Therefore:

$$\begin{aligned} P(\Omega \cup \emptyset) &= 1 && \text{(Use axiom 3:)} \\ P(\Omega) + P(\emptyset) &= 1 && \text{(Apply axiom 2:)} \\ 1 + P(\emptyset) &= 1 \\ P(\emptyset) &= 0 \end{aligned}$$

We have proven property 2. ■

Property 3. $P(A) \leq 1$

Proof. By the first axiom of probability, $P(A^c) \geq 0$. From property 1, we know that $P(A^c) = 1 - P(A)$. Then:

$$\begin{aligned} 0 &\leq P(A^c) \\ 0 &\leq 1 - P(A) \\ P(A) &\leq 1 \end{aligned}$$

This proves property 3. ■

Property 4. *If* $A \subset B$, *then* $P(A) \leq P(B)$

Proof. Because $A \subset B$, $A = A \cap B$. Applying the probability function yields:

$$\begin{aligned} P(A) &= P(A \cap B) \\ P(A) &= P(B) - P(B \cap A^c) \\ P(A) + P(B \cap A^c) &= P(B) \end{aligned}$$

Now we consider two cases. First, we consider the possibility that $P(B \cap A^c) = 0$. Then:

$$P(B) = P(A) \quad \longrightarrow \quad P(B) \geq P(A)$$

Next, we consider the possibility that $P(B \cap Ac) > 0$. Then:

$$P(B) = P(A) + \underbrace{P(B \cap A^c)}_{>0} \quad \longrightarrow \quad P(B) \geq P(A)$$

There is not a third case where $P(B \cap A^c) < 0$ as the probability measure is defined on $[0, 1]$. Therefore, in all cases $P(B) \geq P(A)$. This proves property 4. ■

Property 5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. Note that $A \cup B$ can be rewritten as $A \cup (B \cap A^c)$. Applying the probability function and the third axiom of probabilty gives:

$$A \cup B = A \cup (B \cap A^c)$$
$$P(A \cup B) = P(A) + P(B \cap A^c)$$

Set B can also be rewritten:

$$B = (B \cap A) \cup (B \cap A^c) \qquad \text{(Apply axiom 3:)}$$
$$P(B) = P(B \cap A) + P(B \cap A^c)$$

Now subtract $P(B)$ from $P(A \cup B)$:

$$P(A \cup B) - P(B) = P(A) + P(B \cap A^c) - P(B \cap A) - P(B \cap A^c)$$
$$P(A \cup B) = P(A) + P(B) - P(B \cap A)$$

This proves property 5. ■

Property 6. $P(A \cup B) \leq P(A) + P(B)$

Proof. From property 5, we know that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now consider the case where $P(A \cap B) = 0$. Then:

$$P(A \cup B) = P(A) + P(B) \quad \longrightarrow \quad P(A \cup B) \leq P(A) + P(B)$$

Next, consider the case where $P(A \cap B) > 0$. Then:

$$P(A \cup B) = P(A) + P(B) \underbrace{-P(A \cap B)}_{<0} \quad \longrightarrow \quad P(A \cup B) \leq P(A) + P(B)$$

In all cases, $P(A \cup B) \leq P(A) + P(B)$, proving property 6. ■

Property 7. $P(A \cap B) \geq P(A) + P(B) - 1$

Proof. From property 3, we know that $P(A \cup B) \leq 1$. From property 5, we know that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Combining the two properties yields:

$$P(A) + P(B) - P(A \cap B) \leq 1$$
$$P(A \cap B) \geq P(A) + P(B) - 1$$

We have now proven property 7 ■

Other Proofs

Partitioning Theorem. *If $\{B_1, B_2, ...\}$ is a partition of Ω, then for any event A:*

$$A = \bigcup_{i=1}^{\infty} A \cap B_i$$

and the sets $(A \cap B_i)$ are mutually disjoint.

Proof. I start by taking any element $a \in A$. Because $\{B_1, B_2, ...\}$ is a partition of Ω, a must be an element of B_i for some i. Then, a must be an element of $A \cap B_i$, for some i. But because $A \cap B_i \subset \cup_{i=1}^{\infty} A \cap B_i$, a must also be an element of $\cup_{i=1}^{\infty} A \cap B_i$. Therefore, $a \in A \quad \longrightarrow \quad a \in \cup_{i=1}^{\infty} A \cap B_i$.

Now consider any element $b \in \cup_{i=1}^{\infty} A \cap B_i$. Because B_i is mutually disjoint with all B_j for $i \neq j$, a is an element of a single $A \cap B_i$, for some i. If $a \in A \cap B_i$, a must also be an element of A, by definition of the intersection. Therefore, $a \in \cup_{i=1}^{\infty} A \cup B_i \quad \longrightarrow \quad a \in A$.

Because $a \in A \quad \longrightarrow \quad a \in \cup_{i=1}^{\infty} A \cap B_i$ and $a \in \cup_{i=1}^{\infty} A \cup B_i \quad \longrightarrow \quad a \in A$, we conclude that:

$$A = \bigcup_{i=1}^{\infty} A \cap B_i$$

This proves the first part of the partitioning theorem. To prove the second part, we prove that the intersection of any $A \cap B_i$ and $A \cap B_j$ for $i \neq j$ is empty:

$$\begin{aligned}(A \cap B_i) \cap (A \cap B_j) &= A \cap (B_i \cap B_j) \\ &= A \cap \emptyset \\ &= \emptyset\end{aligned}$$

This proves the second part of the partitioning theorem. ■

Principle of Equally-Likely Outcomes. *If an experiment has N symmetric outcomes $a_1, ...a_N$, then $P(a_i) = \frac{1}{N}$.*

Proof. Define Ω as the space consisting of $\{a_1, ..., a_N\}$. Then assign each outcome to exactly one event (e.g. $a_1 \in A_1$, $a_2 \in A_2$,...). Then $\{A_1, ...A_N\}$ is a partition of Ω. By the third axiom of probability:

$$\begin{aligned}P(\cup_{i=1}^{N} A_i) &= \sum_{i=1}^{N} P(A_i) \\ P(\Omega) &= \sum_{i=1}^{N} P(A_i)\end{aligned}$$

Now impose symmetric outcomes and use the second axiom of probability. Then:

$$\begin{aligned}1 &= NP(A_i) \\ \frac{1}{N} &= P(A_i)\end{aligned}$$

Because each event has exactly one element, the event and its corresponding element are equivalent. Then:

$$P(a_i) = \frac{1}{N}$$

This proves the claim. ■